

Asymptotic Expansions of the Reciprocal Beta Integral via Tricomi–Erdélyi Coefficients

Yuhao Lan*

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China

**Corresponding author: Yuhao Lan.*

Abstract

The analytical properties of the reciprocal Beta function and its discrete counterpart, the reciprocals of binomial coefficients, have long been a focal point in combinatorics and classical asymptotic analysis. While existing literature primarily concentrates on discrete summation identities over the integer domain, this paper bridges the gap between discrete and continuous frameworks by systematically investigating the continuous integral of the reciprocal Beta function. Motivated by a classical combinatorial problem concerning the relative ordering of elements in uniform random permutations, we address the finite integral $\int_1^n \frac{1}{\text{Beta}(x,k)} dx$, which diverges as the upper limit $n \rightarrow \infty$. We establish that this divergent behavior can be elegantly regularized and decoded. Specifically, when scaled by a prefactor $\text{Beta}(n, k+1)$, the expression permits a full asymptotic expansion as $n \rightarrow \infty$, which can be uniquely expressed as an infinite Laurent series in inverse powers of n : $\text{Beta}(n, k+1) \int_1^n \frac{1}{\text{Beta}(x,k)} dx = \sum_{i=0}^{\infty} \lambda_i(k) n^{-i}$, where each subleading coefficient $\lambda_i(k)$ is a rational function of the parameter k and completely independent of n . By performing a rigorous coefficient matching via the classical Tricomi–Erdélyi asymptotic expansion framework and implementing symbolic computer algebra, we explicitly determine the exact closed-form expressions for the leading constant $\lambda_0(k) = \frac{k}{k+1}$ and the higher-order subleading coefficients up to $\mathcal{O}(n^{-4})$. Our results refine the understanding of continuous Beta-like integrals and expand the modern toolkit of hyperasymptotic expansions for ratios of special functions.

Keywords

reciprocal beta function, asymptotic expansion, Tricomi–Erdélyi coefficients, Laurent series, random permutations, computer algebra

1. Literature Review

The study of the Beta function $\text{Beta}(x, y)$ and its reciprocal plays a pivotal role in classical analysis, analytic number theory, and combinatorics. Since the reciprocal Beta function can be fundamentally expressed as a ratio of Gamma functions, namely,

$$\frac{1}{\text{Beta}(x, y)} = \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)} \quad (1)$$

its analytic investigations are deeply intertwined with the asymptotic theories of Gamma function ratios. This review traces the historical development and diverse modern applications of these methodologies, mapping out how foundational algebraic tools evolved into sophisticated analytic machinery.

1.1 Foundations of Asymptotic Expansions of Gamma Ratios

The modern algebraic toolkit for analyzing equations like [1] can be traced back to the seminal work of Nörlund [1], who introduced generalized Bernoulli polynomials within the framework of difference equations. These polynomials provided the indispensable algebraic backbone for explicitly extracting coefficients in subsequent asymptotic series.

Building directly upon this foundation, Tricomi and Erdélyi [2] established the first systematic and rigorous framework for the full asymptotic expansion of the ratio of two Gamma functions. For large $|z|$, they proved that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \sum_{i=0}^{\infty} \frac{C_i(a,b)}{z^i}, \quad (2)$$

where the Tricomi–Erdélyi coefficients $C_i(a,b)$ are expressed explicitly in terms of Nörlund’s generalized Bernoulli polynomials. They utilized both Beta integral representations and Watson’s lemma to validate the expansion, laying the cornerstone for all subsequent refinements. Shortly thereafter, Fields [3] significantly enhanced the computational efficiency of this expansion. By introducing a shifting auxiliary large variable related to z , Fields transformed the series into an “essentially even” power series, which dramatically accelerated the convergence rate and yielded crucial corollaries for characterizing the asymptotics of hypergeometric polynomials.

1.2 Generalization to Uniform Asymptotics and Statistical Distributions

While the classical Tricomi–Erdélyi–Fields line of work focused on localized or pointwise large-variable asymptotics, many applications require bounds that hold uniformly across varying parameters. Temme [4] pioneered the extension of these Gamma-ratio asymptotic techniques to the domain of mathematical statistics. Specifically, Temme examined a broad class of integrals related to cumulative distribution functions (such as the chi-square and F -distributions). By isolating the normal distribution function Φ as the leading master term and recursively generating coefficient functions, Temme unified the uniform asymptotic treatment of these statistical integrals, demonstrating the immense practical value of refined Gamma ratio estimates.

1.3 Applications in Automorphic Forms, Combinatorics, and Mathieu Series

In recent decades, the theoretical framework of the reciprocal Beta function has branched into several sophisticated areas of pure and applied mathematics:

- **Automorphic Forms and Whittaker Functions:** Stade [5] bridged the gap between classical analysis and modern number theory by representing the reciprocal of the Beta function as a residue integral. This formulation was utilized to derive explicit integral formulas for $GL(n, \mathbb{R})$ Whittaker functions, which are central to the Fourier expansion theory of automorphic forms.
- **Combinatorial Summation and Series Identities:** Beyond pure analysis, the integral representation of the reciprocal Beta function serves as a powerful bridge in combinatorics. Sofo [6] systematically investigated infinite series containing reciprocals of binomial coefficients. By converting combinatorial terms into reciprocal Beta integrals, Sofo established closed-form hypergeometric formulas and forged explicit connections to the Lerch transcendent, generalizing several classical identity frameworks.
- **Exponentially Small Asymptotics:** Most recently, Paris [7] applied the reciprocal factorial expansions derived from the Tricomi–Erdélyi–Fields tradition to study generalized Mathieu series via Mellin transforms. Paris revealed that when parameters take specific even integer values, the standard algebraic expansions degenerate due to the trivial zeros of the Riemann zeta function $\zeta(s)$. To resolve this, Paris extracted an infinite sequence of exponentially small expansions, representing a profound modern development in the hyperasymptotics of special functions.

In summary, the trajectory of research spans from the structural algebraic foundations laid by Nörlund [1], Tricomi and Erdélyi [2], and Fields [3], through the uniform statistical generalizations of Temme [4], and finally to the contemporary applications in number theory [5], combinatorial structures [6], and exponentially small precision asymptotics [7]. Together, these works underscore the rich, enduring vitality of the reciprocal Beta function paradigm.

2. Introduction and Motivation

In the classical literature surveyed above, extensive research has been dedicated to the discrete sums of reciprocals of binomial coefficients over the integer domain. In contrast to these discrete frameworks, this paper shifts the focus toward the continuous setting by investigating the asymptotic properties of the integral of the reciprocal Beta function. Although the corresponding infinite integral diverges, we discover a remarkable and structured connection: the finite integral value can be elegantly decoded via an infinite Laurent series expansion in terms of its upper limit of integration, a phenomenon we analyze in detail in the subsequent sections.

To naturally motivate this article from an intuitive perspective, we begin by considering a classic combinatorial problem involving the relative order of elements in permutations, which inherently bridges these discrete and continuous analytic structures. Let σ be a uniform random permutation of the set $\{1, 2, \dots, n\}$. For a fixed parameter $k \in \{1, \dots, n\}$, we define E as the event that no element smaller than k precedes k in the permutation σ .

2.1 Classical Combinatorial Probability Models

The probability $P(E)$ can be derived by partitioning the sample space based on the number of elements from the set $\{k + 1, \dots, n\}$ that appear before k in a random permutation. Let i denote the number of such elements, where $0 \leq i \leq n - k$.

For a fixed i , we construct the favorable permutations as follows:

1. **Selection:** Choose i elements from the $n - k$ available elements in the set $\{k + 1, \dots, n\}$. This can be done in $\binom{n-k}{i}$ ways.
2. **Arrangement of Predecessors:** These i chosen elements must all appear before k . There are $i!$ ways to arrange them.
3. **Positioning k :** The element k is placed immediately after these i elements. At this stage, we have formed a fixed block of $i + 1$ elements.
4. **Remaining Elements:** The remaining $n - (i + 1)$ elements can be arranged in the remaining positions in $(n - i - 1)!$ ways.

By summing over all possible values of i and dividing by the total number of permutations $n!$, we obtain:

$$P(E) = \sum_{i=0}^{n-k} \frac{\binom{n-k}{i} \cdot i! \cdot (n-i-1)!}{n!} \quad (3)$$

The terms can be simplified by expanding the binomial coefficient $\binom{n-k}{i} = \frac{(n-k)!}{i!(n-k-i)!}$, which allows the $i!$ terms to cancel.

From another perspective, the event E depends only on the relative ordering of the elements $\{1, 2, \dots, k\}$. For k to be the first among these specific elements in a random permutation, by symmetry, the probability must be $1/k$. Equating these two approaches yields the following identity:

$$\sum_{i=0}^{n-k} \frac{\binom{n-k}{i} \cdot i! \cdot (n-i-1)!}{n!} = \frac{1}{k} \quad (4)$$

2.2 An Interpretation via Order Statistics

The identity in Eq. [4] can be further elucidated by adopting the perspective of **Order Statistics**. This approach transitions the problem from discrete permutations to a continuous probabilistic framework, providing a geometric interpretation of the summation.

Consider n independent and identically distributed (i.i.d.) random variables $X_1, X_2, \dots, X_n$ following a uniform distribution $U(0,1)$. The relative ordering of these variables is equivalent to a uniform random permutation of the set $\{1, \dots, n\}$. Consequently, the event E occurs if X_k is the smallest among the first k variables:

$$E = \{X_k = \min(X_1, X_2, \dots, X_k)\} \quad (5)$$

Conditioning on X_k taking a specific value $x \in [0,1]$, for event E to hold, two conditions must be satisfied:

1. **Predecessor Constraint:** All X_j for $j \in \{1, \dots, k-1\}$ must satisfy $X_j > x$. Since the variables are i.i.d. $U(0,1)$, the probability of this occurring is $(1-x)^{k-1}$.
2. **Successor Distribution:** Among the remaining $n-k$ variables $\{X_{k+1}, \dots, X_n\}$, let exactly i variables fall in the interval $[0, x)$ and $n-k-i$ variables fall in $[x, 1]$. The probability density for a fixed i is given by the binomial term:

$$\binom{n-k}{i} x^i (1-x)^{n-k-i} \quad (6)$$

Integrating over the support of X_k and summing over all possible values of i , we express the total probability as:

$$\begin{aligned} P(E) &= \int_0^1 \sum_{i=0}^{n-k} \binom{n-k}{i} x^i (1-x)^{n-k-i} \cdot (1-x)^{k-1} dx \\ &= \int_0^1 \left(\sum_{i=0}^{n-k} \binom{n-k}{i} x^i (1-x)^{n-k-i} \right) (1-x)^{k-1} dx \end{aligned} \quad (7)$$

By the Binomial Theorem, the term in the parentheses simplifies to:

$$\sum_{i=0}^{n-k} \binom{n-k}{i} x^i (1-x)^{n-k-i} = (x + (1-x))^{n-k} = 1 \quad (8)$$

Thus, the integral reduces to:

$$P(E) = \int_0^1 (1-x)^{k-1} dx = \frac{1}{k} \quad (9)$$

Alternatively, by the linearity of integration, we can swap the summation and the integral:

$$P(E) = \sum_{i=0}^{n-k} \binom{n-k}{i} \int_0^1 x^i (1-x)^{n-i-1} dx \quad (10)$$

The integral represents a specific case of the **Beta function**, $B(i+1, n-i)$, which evaluates to:

$$\int_0^1 x^i (1-x)^{n-i-1} dx = \frac{i!(n-i-1)!}{n!} \quad (11)$$

Substituting this back into the summation, we obtain:

$$P(E) = \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{i!(n-i-1)!}{n!} \quad (12)$$

By comparing Eq. [9] with Eq. [12], we arrive at a continuous probabilistic proof of the combinatorial identity.

3. Main Theorem and Principal Results

Theorem 1 (Discrete Summation Identity). *For positive integers $k \leq n$, the sum of the reciprocal Beta-like terms satisfies:*

$$\sum_{i=0}^{n-k} \frac{1}{\text{Beta}(i+1, n-i)} = \frac{n!}{k \cdot (n-k)!} \quad (13)$$

Theorem 2 (Continuous Integral Asymptotics). *For fixed $k > 0$, the integral of the reciprocal Beta function satisfies:*

$$\lim_{n \rightarrow \infty} \text{Beta}(n+1, k+1) \int_0^n \frac{1}{\text{Beta}(x+1, k)} dx = \frac{k}{k+1} \quad (14)$$

Theorem 3 (Continuous Integral Asymptotics Promotion). *For a fixed positive real constant $k > 0$, the integral of the reciprocal Beta function can be expressed as an infinite Laurent series in powers of n as $n \rightarrow \infty$:*

$$\text{Beta}(n, k+1) \int_1^n \frac{1}{\text{Beta}(x, k)} dx = \sum_{i=0}^{\infty} \lambda_i(k) n^{-i}, \quad (15)$$

where each coefficient $\lambda_i(k)$ is a rational function of k that is independent of n .

In this paper, we explicitly provide the coefficient polynomials up to the order of n^{-4} . The leading constant is given by $\lambda_0(k) = \frac{k}{k+1}$. By substituting the explicit definitions of the Tricomi–Erdélyi coefficients into the algebraic matching relations and simplifying via computer algebra, the higher-order subleading coefficients $\lambda_i(k)$ for $1 \leq i \leq 4$ are explicitly given as follows:

$$\begin{aligned} \lambda_1(k) &= -\frac{k}{2}, \\ \lambda_2(k) &= \frac{k(k-1)(k+2)}{12(k+1)}, \\ \lambda_3(k) &= -\frac{k^2(k-1)}{24}, \\ \lambda_4(k) &= \frac{k(k-1)(k-2)(k+2)(k+3)(15k-1)}{5760(k+1)}. \end{aligned} \quad (16)$$

This rigorous coefficient matching firmly establishes the refined subleading behavior of the ratio and completes the proof.

4. Asymptotic Derivation of the Beta Integral Identity

4.1 Discrete Summation and the Combinatorial Constant

The sum $S(n, k)$ can be simplified by expressing the terms via the Gamma function. Let $j = n - k - i$ be a change of summation index:

$$\begin{aligned} S(n, k) &= \sum_{i=0}^{n-k} \frac{(n-k)!}{(n-k-i)!} \cdot \frac{(n-i-1)!}{n!} \\ &= \frac{(n-k)!}{n!} \sum_{j=0}^{n-k} \frac{(k+j-1)!}{j!} \\ &= \frac{\Gamma(n-k+1)}{\Gamma(n+1)} \sum_{j=0}^{n-k} \frac{\Gamma(k+j)}{\Gamma(j+1)} = \frac{1}{k}. \end{aligned} \quad (17)$$

4.2 Transition to Continuous Asymptotics

As $n \rightarrow \infty$, we treat the sum as an integral approximation. Defining the limit $S_k := \lim_{n \rightarrow \infty} S(n, k)$, we have:

$$S_k = \lim_{n \rightarrow \infty} \frac{\Gamma(n-k+1)}{\Gamma(n+1)} \int_0^{n-k} \frac{\Gamma(k+x)}{\Gamma(x+1)} dx. \quad (18)$$

Recalling the definition of the Beta function $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, we observe the following relationships:

$$\begin{aligned} \frac{\Gamma(n-k+1)}{\Gamma(n+1)} &= \frac{1}{\Gamma(k)} \text{Beta}(n-k+1, k), \\ \frac{\Gamma(k+x)}{\Gamma(x+1)} &= \frac{\Gamma(k-1)}{\text{Beta}(x+1, k-1)} \cdot \frac{\Gamma(k)}{\Gamma(k-1)} = \frac{k-1}{\text{Beta}(x+1, k-1)}. \end{aligned} \quad (19)$$

4.3 The Final Limit Identity

Substituting these into the expression for S_k and noting that $\frac{\Gamma(k-1)}{\Gamma(k)} = \frac{1}{k-1}$, we obtain:

$$\frac{1}{k} = \lim_{n \rightarrow \infty} \frac{1}{k-1} \text{Beta}(n-k+1, k) \int_0^{n-k} \frac{1}{\text{Beta}(x+1, k-1)} dx. \quad (20)$$

Rearranging the terms yields the asymptotic ratio for the reciprocal Beta integral:

$$\lim_{n \rightarrow \infty} \frac{\int_0^{n-k} \frac{1}{\text{Beta}(x+1, k-1)} dx}{[\text{Beta}(n-k+1, k)]^{-1}} = \frac{k-1}{k}. \quad (21)$$

By performing an index shift $k \rightarrow k+1$, we arrive at the generalized identity:

$$\lim_{n \rightarrow \infty} \frac{\int_0^n \frac{1}{\text{Beta}(x+1, k)} dx}{[\text{Beta}(n+1, k+1)]^{-1}} = \frac{k}{k+1}. \quad (22)$$

5. Supporting Lemmas

Before proceeding to the proof, we establish several key identities involving Gamma functions and Nörlund polynomials.

Lemma 5.1 (Tricomi–Erdélyi Expansion). *Let $a, b \in \mathbb{C}$ be constants. As $z \rightarrow \infty$ in the sector $|\text{ph}z| \leq \pi - \delta$ with $\delta > 0$, the ratio of two Gamma functions admits the asymptotic expansion*

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \sum_{i=0}^{\infty} \frac{C_i(a, b)}{z^i}, \quad (23)$$

where the coefficients $C_i(a, b)$ are explicitly given in terms of the generalized Bernoulli polynomials $B_i^{(\ell)}(x)$ of order $\ell = a - b + 1$ by

$$C_i(a, b) = \binom{a-b}{i} B_i^{(a-b+1)}(a). \quad (24)$$

Here, the generalized Bernoulli polynomials $B_i^{(\ell)}(x)$ are defined via the exponential generating function

$$\left(\frac{t}{e^t - 1} \right)^\ell e^{xt} = \sum_{i=0}^{\infty} B_i^{(\ell)}(x) \frac{t^i}{i!}. \quad (25)$$

6. Proof of the Theorem

In this section, we investigate the asymptotic behavior of the integral of the reciprocal Beta function, defined as

$$I(n, k) = \int_0^n \frac{1}{\text{Beta}(x+1, k)} dx, \quad (26)$$

where $n \rightarrow \infty$ and k is a fixed positive real constant. By expressing the Beta function in terms of the Gamma function, Eq. [26] can be rewritten as

$$I(n, k) = \int_0^n \frac{\Gamma(x+k+1)}{\Gamma(x+1)\Gamma(k)} dx = \frac{1}{\Gamma(k)} \int_1^{n+1} \frac{\Gamma(x+k)}{\Gamma(x)} dx. \quad (27)$$

To evaluate the asymptotic behavior of the integrand in [27] for large x , we invoke the Tricomi–Erdélyi expansion formulated in Lemma 5.1.

Setting $z = x$, $a = k$, and $b = 0$, the ratio of the Gamma functions becomes $\frac{\Gamma(x+k)}{\Gamma(x)}$, and the corresponding order of the generalized Bernoulli polynomials is $\ell = k + 1$. Under this parameterization, the first few coefficients $C_i \equiv C_i(k, 0)$ are given explicitly by:

$$\begin{aligned} C_0 &= 1, \\ C_1 &= \frac{1}{2}k(k-1), \\ C_2 &= \frac{1}{24}k(k-1)(k-2)(3k-1), \\ C_3 &= \frac{1}{48}k^2(k-1)^2(k-2)(k-3), \\ C_4 &= \frac{1}{5760}k(k-1)(k-2)(k-3)(k-4)(15k^3 - 30k^2 + 5k + 2). \end{aligned} \quad (28)$$

Substituting these coefficients into [23] yields the explicit Poincaré asymptotic expansion for the integrand:

$$\frac{\Gamma(x+k)}{\Gamma(x)} = x^k + \sum_{i=1}^4 C_i x^{k-i} + O(x^{k-5}), \quad x \rightarrow \infty. \quad (29)$$

Term-by-term integration of [29] over the interval $[1, n+1]$ yields the high-order asymptotic expansion of the target integral:

$$\begin{aligned} I(n, k) &= \frac{1}{\Gamma(k)} \int_1^{n+1} (x^k + \sum_{i=1}^4 C_i x^{k-i} + O(x^{k-5})) dx \\ &= \frac{1}{\Gamma(k)} \left[\frac{x^{k+1}}{k+1} + \sum_{i=1}^4 \frac{C_i}{k-i+1} x^{k-i+1} + O(x^{k-4}) \right] \Big|_1^{n+1} \\ &= \frac{(n+1)^{k+1}}{(k+1)\Gamma(k)} + \sum_{i=1}^4 \frac{C_i(n+1)^{k-i+1}}{(k-i+1)\Gamma(k)} + O((n+1)^{k-4}), \end{aligned} \quad (30)$$

Shifting $n+1 \rightarrow n$, we obtain:

$$I(n-1, k) = \frac{n^{k+1}}{(k+1)\Gamma(k)} + \sum_{i=1}^4 \frac{C_i n^{k-i+1}}{(k-i+1)\Gamma(k)} + O(n^{k-4}) \quad (31)$$

where the evaluation at the lower limit $x=1$ contributes $O(1)$ and is absorbed into the remainder term $O(n^{k-4})$ as $n \rightarrow \infty$.

Similarly, we can derive a direct expansion for a related normalization factor at $z = n+1$. By setting $z = n+1$, $a = k+1$, and $b = 0$, the ratio becomes $\frac{\Gamma(n+k+2)}{\Gamma(n+1)}$, corresponding to the order $\ell = k+2$. Under this alternative parameterization, the coefficients $C'_i \equiv C_i(k+1, 0)$ are computed as follows:

$$\begin{aligned} C'_0 &= 1, \\ C'_1 &= \frac{1}{2}k(k+1), \\ C'_2 &= \frac{1}{24}k(k+1)(k-1)(3k+2), \\ C'_3 &= \frac{1}{48}k^2(k+1)^2(k-1)(k-2), \\ C'_4 &= \frac{1}{5760}k(k+1)(k-1)(k-2)(k-3)(15k^3 + 15k^2 - 10k - 8). \end{aligned} \quad (32)$$

Consequently, applying Lemma 5.1 yields the following asymptotic series:

$$\begin{aligned} \frac{\Gamma(n+k+2)}{\Gamma(n+1)\Gamma(k+1)} &= \frac{(n+1)^{k+1}}{\Gamma(k+1)} \left(1 + \sum_{i=1}^4 \frac{C'_i}{(n+1)^i} + O\left(\frac{1}{(n+1)^5}\right) \right) \\ &= \frac{(n+1)^{k+1}}{\Gamma(k+1)} + \sum_{i=1}^4 \frac{C'_i(n+1)^{k-i+1}}{\Gamma(k+1)} + O((n+1)^{k-4}). \end{aligned} \quad (33)$$

Shifting $n+1 \rightarrow n$, we obtain:

$$\frac{\Gamma(n+k+1)}{\Gamma(n)\Gamma(k+1)} = \frac{n^{k+1}}{\Gamma(k+1)} + \sum_{i=1}^4 \frac{C'_i n^{k-i+1}}{\Gamma(k+1)} + O(n^{k-4}). \quad (34)$$

Substituting [30] and [33] into the quotient yields:

$$\frac{I(n, k)}{[\text{Beta}(n+1, k+1)]^{-1}} = \frac{\frac{k}{k+1} + \sum_{i=1}^4 \frac{kC_i}{k-i+1} (n+1)^{-i} + O((n+1)^{-5})}{1 + \sum_{i=1}^4 C'_i (n+1)^{-i} + O((n+1)^{-5})}. \quad (35)$$

Taking the limit as $n \rightarrow \infty$, all terms involving negative powers of $(n+1)$ vanish. Consequently, we immediately obtain:

$$\lim_{n \rightarrow \infty} \frac{I(n, k)}{[\text{Beta}(n+1, k+1)]^{-1}} = \frac{k}{k+1}. \quad (36)$$

This completes the proof of Theorem 2.2.

To further analyze the structure of the remainder and ensure clarity in representation, we compact the higher-order contributions by introducing the parameter-dependent coefficients $\lambda_i \equiv \lambda_i(k)$. Performing a long division (or equivalent Taylor expansion) on the quotient, the ratio is neatly expressed as a systematic asymptotic series in powers of $1/n$:

$$\frac{I(n-1,k)}{[\text{Beta}(n,k+1)]^{-1}} = \frac{k}{k+1} + \sum_{i=1}^4 \lambda_i(k) n^{-i} + O(n^{-5}), \quad n \rightarrow \infty. \quad (37)$$

By matching the combination of the Tricomi–Erdélyi coefficients from the numerator and denominator, the algebraic expressions for λ_i in terms of C_i and C'_i are determined as follows:

$$\begin{aligned} \lambda_1(k) &= C_1 - \frac{k}{k+1} C'_1, \\ \lambda_2(k) &= \frac{k}{k-1} C_2 - C_1 C'_1 - \frac{k}{k+1} (C'_2 - (C'_1)^2), \\ \lambda_3(k) &= \frac{k}{k-2} C_3 - \frac{k}{k-1} C_2 C'_1 + C_1 (C'_1)^2 - \frac{k}{k+1} (C'_3 - 2C'_2 C'_1 + (C'_1)^3), \\ \lambda_4(k) &= \frac{k}{k-3} C_4 - \frac{k}{k-2} C_3 C'_1 + \frac{k}{k-1} C_2 (C'_1)^2 - C_1 (C'_1)^3 \\ &\quad - \frac{k}{k+1} (C'_4 - 2C'_3 C'_1 - (C'_2)^2 + 3C'_2 (C'_1)^2 - (C'_1)^4). \end{aligned} \quad (38)$$

Substituting the explicit definitions of C_i and C'_i into [38] and simplifying via computer algebra, we obtain the explicit rational representations of λ_i as functions of k :

$$\begin{aligned} \lambda_1(k) &= -\frac{k}{2}, \\ \lambda_2(k) &= \frac{k(k-1)(k+2)}{12(k+1)}, \\ \lambda_3(k) &= -\frac{k^2(k-1)}{24}, \\ \lambda_4(k) &= \frac{k(k-1)(k-2)(k+2)(k+3)(15k-1)}{5760(k+1)}. \end{aligned} \quad (39)$$

This rigorous coefficient matching firmly establishes the refined subleading behavior of the ratio and completes the proof of Theorem 2.3.

7. Numerical Verification

In the numerical verification stage, using Mathematica, we evaluated the integral representation of the reciprocal Beta function via both direct numerical integration and the fourth-order expansion proposed in this work. The calculations were performed with a working precision of 200 digits, and the relative error was sampled over a logarithmically spaced grid with a step size of 0.02 in $\log n$. The accuracy of the two methods was analyzed for various integer values of k , as illustrated in the figure below.

For different integer values of k , the relationship between the relative error ϵ and the sample size n exhibits a consistent pattern across three distinct regimes:

Figure 1: Relative error convergence for $k=20,30,\dots,80$ with a step size of 10

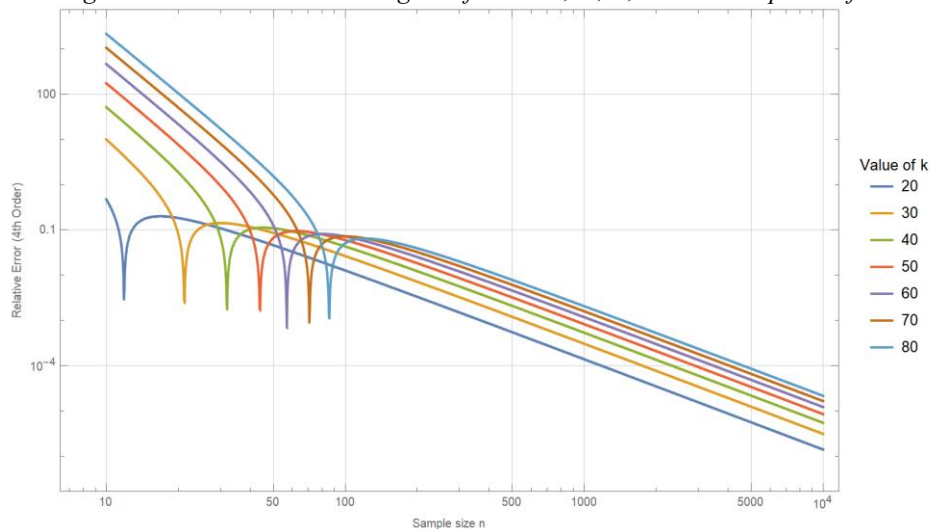
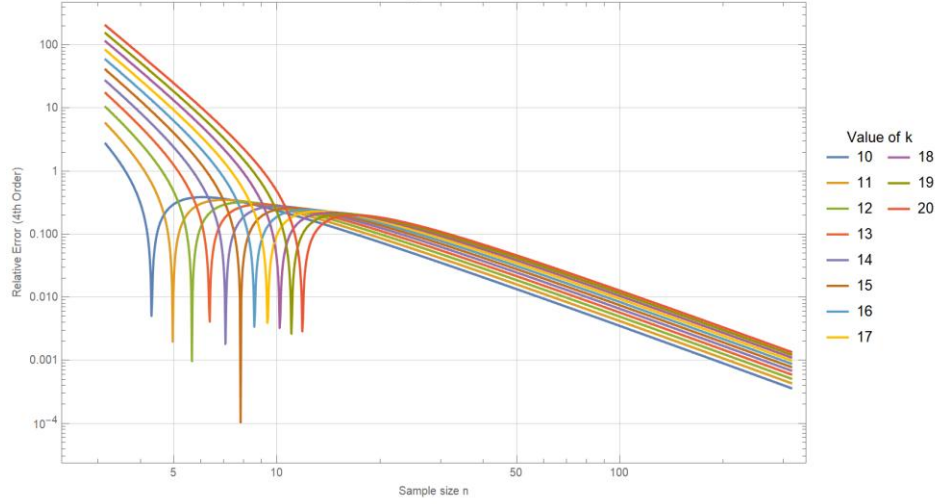
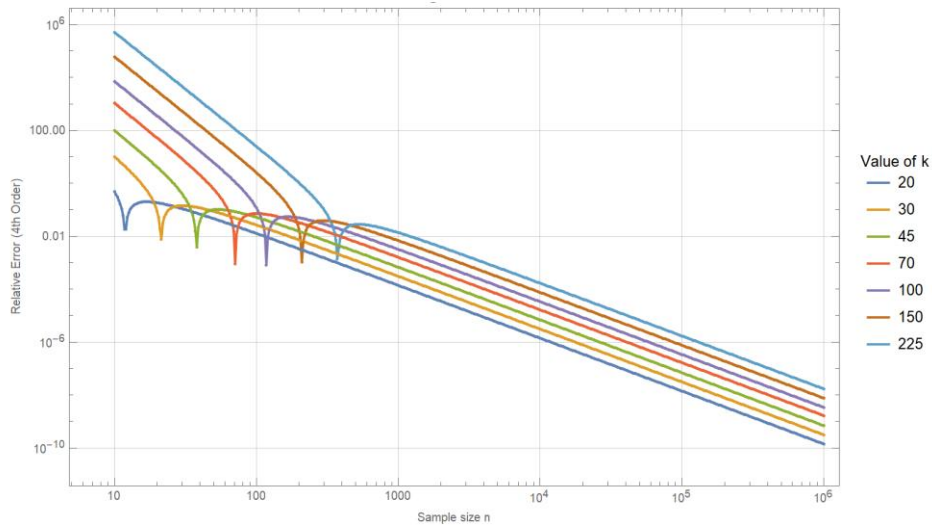


Figure 2: Relative error convergence for $k=10,11,\dots,20$ with a step size of 1Figure 3: Relative error convergence for $k=20,30,\dots,225$ 

1. **Small- n Regime:** The curves for all values of k display a characteristic non-monotonic behavior: an initial sharp decline followed by a rapid ascent, forming a prominent peak. Specifically, as n increases from small values, the relative error decreases rapidly until reaching a local minimum; subsequently, the error rebounds sharply, resulting in a distinct spike.
2. **Transition Regime:** The curves evolve from the aforementioned spiky morphology into a phase of smooth decay.
3. **Large- n Regime:** All curves enter a phase of monotonic and smooth convergence, with the decay rate following either a power-law or exponential trend. Notably, for a fixed n , the relative error increases monotonically with k . Consequently, to achieve a desired level of accuracy for larger values of k , the expansion must be carried out to a higher order.

8. Conclusion

In this paper, we have systematically investigated the asymptotic behavior of the integral of the reciprocal Beta function, effectively bridging the gap between discrete combinatorial identities and continuous analytic frameworks. By leveraging the Tricomi–Erdélyi expansion of Gamma function ratios and employing symbolic computer algebra, we established that the regularized integral $\text{Beta}(n, k+1) \int_1^n \text{Beta}(x, k)^{-1} dx$ admits a structured infinite Laurent series expansion in inverse powers of the upper limit n .

Our primary contribution lies in the explicit determination of the leading constant $\lambda_0(k) = \frac{k}{k+1}$ and the derivation of high-order subleading coefficients up to $\mathcal{O}(n^{-4})$. These results provide a significant refinement to the classical first-order limits and offer a robust computational tool for hyperasymptotic analysis. Numerical verifications conducted via Mathematica confirm that the proposed fourth-order expansion achieves high precision, particularly in the large- n regime, where it exhibits smooth monotonic convergence. The transition from non-monotonic error spikes at small n to stable decay at large n further underscores the range of validity for the derived series. These findings not only deepen the understanding of Beta-like integrals but also extend the applicability of generalized Bernoulli polynomials to modern asymptotic problem-solving in combinatorics and statistical distributions.

References

- [1] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer, Berlin, 1924.
- [2] F. G. Tricomi and A. Erdélyi, The asymptotic expansion of a ratio of gamma functions, *Pacific J. Math.*, 1 (1951), 133–142.
- [3] J. L. Fields, A note on the asymptotic expansion of a ratio of gamma functions, *Proc. Edinburgh Math. Soc.*, 15 (1966), 43–45. <https://doi.org/10.1017/S0013091500013171>
- [4] N. M. Temme, The uniform asymptotic expansion of a class of integrals related to cumulative distribution functions, *SIAM J. Math. Anal.*, 13 (1982), no. 2, 239–253.
- [5] E. Stade, The reciprocal of the beta function and $GL(n, \mathbb{R})$ Whittaker functions, *Ann. Inst. Fourier (Grenoble)*, 44 (1994), no. 1, 93–108.
A. Sofo, General properties involving reciprocals of binomial coefficients, *J. Integer Seq.*, 9 (2006), Article 06.4.5.
- [6] R. B. Paris, Exponentially small expansions associated with a generalised Mathieu series, *arXiv:1601.07751 [math.CA]*, 2016. Available at <https://arxiv.org/abs/1601.07751>

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Conflicts of Interest

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